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The cohomological supercharge

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Abstract: We discuss the supersymmetry operator in the cohomological formulation of dimensionally reduced SYM. By establishing the cohomology, a large class of invariants are classified.

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Contents

1. Introduction

A deep relation between dimensionally reduced supersymmetric Yang-Mills theory, and cohomological field theory was first uncovered by Moore, Nekrasov and Shatashvili [1]. To illustrate, we shall consider the four-dimensional euclidean SYM, although the techniques which follow can be applied immediately to the cases of six and ten dimensions. After reduction to zero dimensions, the action is

$$S^0_{\mathrm{E}} = -\operatorname{Tr}\left(\frac{1}{4}[X_\mu, X_\nu]^2 + \overline{\lambda}\overline{\sigma}^\mu[X_\mu, \lambda]\right). \tag{1.1}$$

All fields are $N \times N$ matrices and transform in the adjoint representation of the gauge group $G = \mathrm{SU}(N)/\mathbb{Z}_N$. The gauge fields X_μ ($\mu = 1, \ldots, 4$) are restricted to the Lie algebra of G which is the set of traceless hermitean matrices. The fermions λ are complex traceless grassman matrices. The σ^i are the Pauli matrices and $\sigma^4 = i\,1_2$.

A quantity of particular interest is the matrix integral

$$I_{4,N} = \int dX\, d\lambda dD \exp \operatorname{Tr}\left(\frac{1}{4}[X_\mu, X_\nu]^2 + \overline{\lambda}\overline{\sigma}^\mu[X_\mu, \lambda] - 2D^2\right) \tag{1.2}$$

(here an auxiliary field D with appropriate integration measure has been added). In the case of ten dimensions, this integral is of relevance in its own right as a D-instanton partition function [2], and in a conjectured non-perturbative description of IIB superstring theory [3]. It also arises as the principle part of the Witten Index in supersymmetric Yang-Mills quantum mechanics [4, 5]. So it is of relevance to counting threshold bound states of D-particles and, when $N \to \infty$, vacuum states in the matrix model of M-theory.

Values for $I_{d,N}$ have been calculated analytically for $N = 2$ [4, 5, 6] and numerically for other small N [7, 8]. In the case of $d = 10$, these results correspond to a conjecture of Green and Gutperle [9].

The approach of [1] is to make the following field replacements. First rewrite the fermions in terms of their hermitean and antihermitean parts

$$\begin{aligned}\lambda_1 &= (\eta_2 + i\eta_1)\\ \lambda_2 &= (\psi_1 + i\psi_2)\end{aligned} \tag{1.3}$$

and the auxiliary field

$$D = H + \frac{1}{2}[X_1, X_2]\,. \tag{1.4}$$

By the usual contour shifting argument for a gaussian integral, H can be taken hermitean. To obtain the related cohomological action, make the replacement

$$\begin{aligned}\phi &= \frac{1}{2}(X_3 + iX_4)\\ \overline{\phi} &= -\frac{1}{2}(X_3 - iX_4)\end{aligned} \tag{1.5}$$

and take $\overline{\phi}$ hermitean and ϕ antihermitean. This gives

$$\begin{aligned}S_E^0 \to S_{\rm coh} = \mathrm{Tr}\Big(&H^2 + H[X_1, X_2] - \epsilon^{ab}\eta_1[\psi_a, X_b] + \eta_2[\psi_a, X_a] -\\ &- \eta_a[\phi, \eta_a] - \psi_a[\overline{\phi}, \psi_a] + [X_a, \phi][X_a, \overline{\phi}] + [\phi, \overline{\phi}]^2\Big)\,.\end{aligned} \tag{1.6}$$

The key point is that ϕ and $\overline{\phi}$ are taken independent. Then $S_{\rm coh}$ describes the dimensional reduction of a Minkowski theory (with lightcone coordinates) whilst S_E^0 gives the ordinary euclidean version.

One can easily write the four linearly independent supercharges of S_E^0 in terms of the new variables, and one of these is

$$\begin{aligned}\delta X_a &= \psi_a & \delta\psi_a &= [\phi, X_a]\\ \delta\overline{\phi} &= -\eta_2 & \delta\eta_2 &= -[\phi, \overline{\phi}]\\ \delta\eta_1 &= H & \delta H &= [\phi, \eta_1]\\ \delta\phi &= 0\,.\end{aligned} \tag{1.7}$$

Since $\delta^2 = [\phi, \ \ ]$, δ is nilpotent on gauge invariant quantities.

The action is δ-exact. $S = \delta Q$, where

$$Q = \mathrm{Tr}\left(\eta_1[X_1, X_2] + \eta_1 H - \psi_a[X_a, \overline{\phi}] - \eta_2[\phi, \overline{\phi}]\right) \tag{1.8}$$

as can readily be checked. So the symmetry $\delta S = 0$ is manifest. The term cohomological to describe the theory is arrived at by analogy of δ with an exterior derivative.

In [1], mass terms are added to the cohomological action whilst preserving a deformed version of δ. This allows the cohomological version of the matrix integral

$$\int d\phi dX_a d\psi_a d\eta_a d\overline{\phi} dH e^{-S_{\rm def}} \tag{1.9}$$

to be calculated. Finally, the matrix integrals over ϕ are replaced with contour integrals.

Remarkably, following this prescription gives precisely the same values as conjectured and obtained numerically for the Yang-Mills integral (1.2). There is recent evidence that the prescription also works for some groups other than SU(N) [10, 11]. An alternative approach also using the cohomological model is given in [12].

In this note, we study the supersymmetry operator δ, and in particular we address the question of which quantities are supersymmetric under δ. Gauge invariant quantities are formed from traces, and so we seek the general solution to the equation

$$\delta \,\mathrm{Tr}\, P = 0\,, \tag{1.10}$$

where P is a polynomial in the matrix fields. In the analogy of δ with an exterior derivative, this is the question of finding the cohomology. An important example of the use is to find the possible supersymmetric deformations of a given action. One usually requires a result valid for any gauge group SU(N), so we shall allow ourselves to make the assumption that N is suitably large. We shall show that

$$\delta \,\mathrm{Tr}\, P = 0 \Leftrightarrow \mathrm{Tr}\, P = \delta \,\mathrm{Tr}\, Q + \mathrm{Tr}\, R(\phi) \tag{1.11}$$

as long as the degree of P is less than $2N/3$.

The proof requires a number of steps. We form a vector space from the polynomials, and deal with issues of linear dependence in section 2.1. A major technical difficulty is that linearly independent polynomials become dependent after applying a trace. This is overcome in section 2.2 by forming a suitable quotient space. Then in section 2.3 the result is proved for a simplified version of δ in which the $[\phi,\ ]$ terms are absent. Finally, in section 2.4, the strands are drawn together to prove the result.

2. Cohomology

2.1 Polynomials

We wish to form a vector space from the polynomials, and eventually argue by induction on degree. However, there is a technical difficulty. Two polynomials which *look* different, because they contain different strings of matrices multiplied together, can turn out to be identical. At this stage, let us be definite and make some careful definitions.

String. A string of length l is a map from $\{1,\ldots,l\}$ into the set of matrix fields. For example, a typical string of length 5 might be

$$\begin{array}{rccccc} s = & X_1 & \eta_2 & \overline{\phi} & X_1 & \phi\,. \\ & (1) & (2) & (3) & (4) & (5) \end{array}$$

Monomial. A monomial of degree $d > 0$ is the matrix product of d matrix fields. The monomial of degree 0 is the identity matrix. For example, a typical monomial of degree 5 might be

$$m = X_1 \cdot \eta_2 \cdot \overline{\phi} \cdot X_1 \cdot \phi\,,$$

where $\cdot$ indicates matrix multiplication.

Each string of length l is naturally associated to a monomial of degree l by applying matrix multiplication between adjacent fields in the string.

Polynomial. A polynomial of degree d is a linear combination of a finite number of monomials whose highest degree is d.

One can form an abstract vector space V_s over $\mathbb{C}$ by taking the strings as the basis. In V_s, the strings are linearly independent. However, as polynomials, the strings are not necessarily linearly independent. This is most easily seen when the matrix size N is 1 so that bosonic matrices commute. Then the two independent strings $X_1\,X_2$ and $X_2\,X_1$ are identical as polynomials. Even when $N > 1$ so that matrices do not commute, it is possible for independent strings to be linearly dependent as polynomials. A trivial example is that $\psi^{N^2} \equiv 0$ when ψ is a traceless hermitean $N \times N$ fermion.

This problem can be overcome by considering only polynomials of degree smaller than the matrix size. Assume that the matrix fields are $N \times N$ and hermitean. They may also have the constraint of tracelessness, but no other constraints. Then the strings of length less than N are linearly independent as polynomials.

To see this, denote the strings of length less than N by $\{s^b\}$ and the corresponding monomials $\{m^b\}$.

Suppose

$$\lambda^b m^b \equiv 0\,, \tag{2.1}$$

for some $\lambda^b \in \mathbb{C}$ and (without loss of generality) $\lambda^1 \neq 0$.

Write $m^1 = Y^1 \cdots Y^d$ where the Y^i are matrix fields and the degree of m^1 is $d < N$.

Then, in particular, the term $Y^1_{12} Y^2_{23} \cdots Y^d_{d,d+1}$ is absent from (2.1). But the only monomial which gives rise to this term is $m^1 = Y^1 \cdots Y^d$. Therefore $\lambda^1 = 0$, and this is a contradiction.

Note that no assumptions are made about which of the matrix fields are fermionic and which bosonic.

2.2 The trace

It will be convenient to work with polynomials rather than traces of polynomials. Unfortunately, two independent polynomials can have identical trace. Define an equivalence relation $P \sim Q \Leftrightarrow \mathrm{Tr}\, P = \mathrm{Tr}\, Q$. Then we would like to form the quotient space $V_p/{\sim}$, where V_p is the vector space of polynomials.

Consider a polynomial $P(A^a)$, where $\{A^a : a = 1, \ldots, M\}$ are the matrix fields. Assume that the only constraints which may be applied to the fields are hermiticity and tracelessness. Define an ordering $\mathcal{O}$ such that

- $\mathcal{O}$ acts individually on each monomial term in P;
- $\mathcal{O}$ cyclically permutes each monomial in P into a preferred form with a sign to respect fermion statistics.

An example of such an ordering would be to define $A^1 > A^2 > A^3 > \cdots$. Then, for example, $\mathcal{O}(A^4A^2A^5) = (-1)^{F_4(F_2+F_5)}A^2A^5A^4$ where F_a is the fermion number of A^a.

Then, for $\deg(P) < N$

$$\mathrm{Tr}\, P = 0 \Leftrightarrow \mathcal{O}(P) = 0 \tag{2.2}$$

so that $\mathcal{O}$ gives a mapping to the quotient space.

To see that (2.2) is true, first note that $\mathcal{O}P = 0 \Rightarrow \mathrm{Tr}\, \mathcal{O}P = 0 \Rightarrow \mathrm{Tr}\, P = 0$ by the cyclic property of trace.

Conversely, suppose $\mathcal{O}P \neq 0$. Consider a particular monomial term in $\mathcal{O}P$:

$$\mathcal{O}P = \lambda Y^1 \cdots Y^M + \cdots\,,$$

where λ is some non-zero coefficient. Then $\mathrm{Tr}\, P = \mathrm{Tr}\, \mathcal{O}P$ contains the term:

$$\mathrm{Tr}\, P = \lambda Y^1_{12} Y^2_{23} \cdots Y^M_{M1} + \cdots\,.$$

Since $\mathcal{O}P$ is ordered, the only monomial that can give such a term is $Y^1Y^2\cdots Y^M$. Therefore this term cannot be cancelled and so $\mathrm{Tr}\, P \neq 0$. Deduce $\mathrm{Tr}\, P = 0 \Rightarrow \mathcal{O}P = 0$.

An ordering operator $\mathcal{O}$ is not the most useful way of dealing with the trace. Since there is no way that $\mathcal{O}$ will commute with any form of supersymmetry operator, it is more helpful to use the following:

Let P_M be a polynomial in which all of the terms are of degree M. Then for $M < N$

$$\mathrm{Tr}\, P_M = 0 \Leftrightarrow \sum_{\substack{\text{cyclic}\\ \text{perms}}} P_M = 0\,. \tag{2.3}$$

The cyclic permutations act on the matrix fields in the monomials, and include a sign to respect fermion statistics. For example, suppose $P = \lambda F_1F_2B_3 + \mu B_1B_2B_4$

where the F_i are fermionic and the B_i bosonic matrix fields. Then the cyclic permutation $\sigma = (123)$ acts as

$$\sigma P = -\lambda F_2 B_3 F_1 + \mu B_2 B_4 B_1 \,.$$

Equation (2.3) follows easily from (2.2). First note that

$$\sum_{\substack{\text{cyclic}\\ \text{perms}}} P_M = \sum_{\substack{\text{cyclic}\\ \text{perms}}} \mathcal{O} P_M \,.$$

Then

$$\operatorname{Tr} P_M = 0 \Rightarrow \mathcal{O} P_M = 0 \Rightarrow \sum_{\substack{\text{cyclic}\\ \text{perms}}} \mathcal{O} P_M = 0 \Rightarrow \sum_{\substack{\text{cyclic}\\ \text{perms}}} P_M = 0 \,.$$

Conversely

$$\operatorname{Tr} \sum_{\substack{\text{cyclic}\\ \text{perms}}} P_M = M \operatorname{Tr} P_M$$

and so

$$\sum_{\substack{\text{cyclic}\\ \text{perms}}} P_M = 0 \Rightarrow \operatorname{Tr} P_M = 0 \,.$$

2.3 Decomposition of the supercharge

If the expression for δ (1.7) did not contain the commutator terms, our task to classify the supersymmetric quantities would be much simpler. In this section, we decompose the supercharge into two parts, and prove our result for the simpler part. This will allow us to tackle the full supercharge in the next section.

Returning to the specific theory under discussion, let us write

$$\begin{aligned} &A^1 = X_1\,, && A^2 = X_2\,, && A^3 = \overline{\phi}\,, && A^4 = \eta_1 \\ &B^1 = \psi_1\,, && B^2 = \psi_2\,, && B^3 = -\eta_2\,, && B^4 = H\,. \end{aligned} \tag{2.4}$$

Then the supersymmetry δ can be written as

$$\begin{aligned} \delta A^i &= B^i\,, \qquad \delta B^i = [\phi, A^i] \\ \delta\phi &= 0\,. \end{aligned} \tag{2.5}$$

If one considers also the six- and ten-dimensional cohomological matrix models, one finds an identical form for δ [1], and so the results from this point on apply equally to all three theories.

Define two new operators d, Δ by

$$\begin{aligned} dA^i &= B^i\,, \qquad dB^i = 0 \\ d\phi &= 0 \\ \Delta A^i &= 0\,, \qquad \Delta B^i = [\phi, A^i] \\ \Delta\phi &= 0 \end{aligned} \tag{2.6}$$

One can very easily check that the following relations hold on polynomials

$$\begin{array}{rr@{\;}l} \text{i)} & \delta^2 & = [\phi, \;\;] \\ \text{ii)} & d^2 & = 0 \\ \text{iii)} & \Delta^2 & = 0 \\ \text{iv)} & \delta & = d + \Delta \\ \text{v)} & \{d, \Delta\} & = \delta^2 = [\phi, \;\;] \end{array} \tag{2.7}$$

and that the operator d has the useful property

$$d \sum_{\substack{\text{cyclic} \\ \text{perms}}} P_M = \sum_{\substack{\text{cyclic} \\ \text{perms}}} dP_M \,, \tag{2.8}$$

which will allow us to deal with the trace.

The first step is to prove the result for a polynomial. Let P be a polynomial of degree less than N, and suppose $dP = 0$. Then

$$dP = 0 \Rightarrow P = dQ + R(\phi) \,, \tag{2.9}$$

for some polynomials Q and R.

To show (2.9), use induction on the degree of P.

The case $\deg(P) = 0$ is simple since then $P = \lambda I = R(\phi)$.

When $\deg(P) > 0$, expand

$$P = A^i S^i + B^i T^i + \phi U + \lambda I \,, \tag{2.10}$$

for some polynomials S^i, T^i and U, and a constant λ. Apply d:

$$0 = dP = B^i S^i + (-1)^{A^i} A^i dS^i + (-1)^{B^i} B^i dT^i + \phi dU \,. \tag{2.11}$$

The notation $(-1)^{A^i}$ is shorthand for ± 1 respectively as A^i is bosonic or fermionic. Then in particular, since d maps bosons to fermions, $(-1)^{B^i} = (-1)^{A^i+1}$.

Since $\deg(P) < N$, the strings are linearly independent (section 2.1). Deduce

$$S^i + (-1)^{A^i+1} dT^i = 0 \tag{2.12}$$

$$dS^i = 0 \tag{2.13}$$

$$dU = 0 \,. \tag{2.14}$$

Note since $d^2 = 0$, (2.13) is implied by (2.12).

By induction, $dU = 0$ implies

$$U = dV + W(\phi) \,, \tag{2.15}$$

where V and W are polynomials.

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Substituting (2.12) and (2.15) into (2.10),

$$\begin{aligned} P &= A^i(-1)^{A^i} dT^i + B^i T^i + \phi\,(dV + W(\phi)) + \lambda I \\ &= d(A^i T^i + \phi V) + \phi W(\phi) + \lambda I \\ &= dQ + R(\phi) \end{aligned} \tag{2.16}$$

and the result (2.9) follows.

Finally in this section, we introduce the trace. Let P be a polynomial of degree less than N, and suppose $d\,\mathrm{Tr}\,P = 0$. Then

$$d\,\mathrm{Tr}\,P = 0 \Rightarrow \mathrm{Tr}\,P = d\,\mathrm{Tr}\,Q + \mathrm{Tr}\,R(\phi)\,, \tag{2.17}$$

for some polynomials Q and R.

To see this, write $P = P_0 + \cdots + P_M$ where each P_i contains only monomials of degree i.

Then, since d preserves the degree of monomials,

$$\begin{aligned} d\,\mathrm{Tr}\,P = 0 &\Rightarrow d\,\mathrm{Tr}\,P_i = 0 \\ &\Rightarrow \mathrm{Tr}\,dP_i = 0 \qquad (i = 0, \ldots, M)\,. \end{aligned}$$

The case of $i = 0$ is simple. For $i > 0$, using (2.3) gives

$$\sum_{\substack{\text{cyclic}\\ \text{perms}}} dP_i = 0$$

and (2.8) implies

$$d \sum_{\substack{\text{cyclic}\\ \text{perms}}} P_i = 0\,.$$

Then (2.9) gives

$$\sum_{\substack{\text{cyclic}\\ \text{perms}}} P_i = dQ_i + R_i(\phi)$$

for some polynomials Q_i and R_i, so that

$$\mathrm{Tr}\,P_i = \frac{1}{i}\,\mathrm{Tr}(dQ_i + R_i(\phi))$$

by the cyclic property of trace.

Then summing over i gives the result.

2.4 Extension to the full supercharge

The task now is to extend the result from d to δ. The commutator terms in the definition of δ make it much harder to deal with the trace. Specifically, δ does not commute with the sum over cyclic permutations. Instead, we proceed with a less direct approach, and make use of the result for d.

Begin with a technical result:

Suppose P_k is a polynomial of degree k satisfying $d\Delta\,\mathrm{Tr}\,P_k = 0$. Then there exists a polynomial Q such that $d\,\mathrm{Tr}\,P_k = (d+\Delta)\,\mathrm{Tr}\,Q$, as long as $N > 3k/2$.

The proof follows an inductive argument. By (2.17),

$$d\Delta\,\mathrm{Tr}\,P_k = 0 \qquad \Rightarrow \qquad \Delta\,\mathrm{Tr}\,P_k = -d\,\mathrm{Tr}\,P_{k+1} + R_{k+1}(\phi)\,,$$

for some polynomials P_{k+1} and R_{k+1} of degree $k+1$. Since neither d nor Δ can produce monomials only in ϕ, $R_{k+1}(\phi) = 0$. Then

$$d\,\mathrm{Tr}\,P_k = (d+\Delta)\,\mathrm{Tr}\,P_k + d\,\mathrm{Tr}\,P_{k+1}\,.$$

On any monomial, d acts to increase the number of fields of type B^i by 1, whilst Δ acts to decrease the number of B^i by 1.

Let M_k be the maximum number of B^i occurring in any term of P_k. Then since $\Delta\,\mathrm{Tr}\,P_k = d\,\mathrm{Tr}\,P_{k+1}$, we have

$$M_{k+1} = M_k - 2\,.$$

Proceed inductively to find

$$d\,\mathrm{Tr}\,P_k = (d+\Delta)\,\mathrm{Tr}(P_k + P_{k+1} + \cdots + P_{k+q}) + d\,\mathrm{Tr}\,P_{k+q+1}\,,$$

where P_{k+q+1} contains no B^i fields at all. Then $\Delta\,\mathrm{Tr}\,P_{k+q+1} = 0$ and so

$$d\,\mathrm{Tr}\,P_k = (d+\Delta)\,\mathrm{Tr}(P_k + \cdots + P_{k+q+1})\,,$$

which proves the result as long as $N > k+q+1$ so that each inductive step is valid. Noting that the case $M_k = k$ is special and can be reduced to the case $M_k = k-1$, one finds that $N > 3k/2$ is a sufficient condition.

We are now ready to prove the main result.

Suppose the matrix fields are of size N, and P is a polynomial in the matrix fields. Then for $\deg(P) < 2N/3$,

$$\delta\,\mathrm{Tr}\,P = 0 \Leftrightarrow \mathrm{Tr}\,P = \delta\,\mathrm{Tr}\,Q + \mathrm{Tr}\,R(\phi)\,, \tag{2.18}$$

where Q and R are polynomials.

To show this, write $P = P_0 + \cdots + P_M$ where P_i contains monomials only of degree i. Then

$$\delta\,\mathrm{Tr}\,P = 0 \qquad \Rightarrow \qquad (d+\Delta)\,\mathrm{Tr}\,P = 0$$

and since d preserves degree whilst Δ increases degree by 1, we have

$$\begin{aligned} \Delta\,\mathrm{Tr}\,P_M &= 0 \\ \Delta\,\mathrm{Tr}\,P_i + d\,\mathrm{Tr}\,P_{i+1} &= 0, \qquad i = 0,\ldots,M-1 \\ d\,\mathrm{Tr}\,P_0 &= 0\,. \end{aligned}$$

By (2.17),

$$d\,\mathrm{Tr}\,P_0 = 0 \qquad \Rightarrow \qquad \mathrm{Tr}\,P_0 = d\,\mathrm{Tr}\,Q_0 + \mathrm{Tr}\,R_0(\phi)$$

and

$$\begin{aligned}\Delta\,\mathrm{Tr}\,P_0 + d\,\mathrm{Tr}\,P_1 = 0 &\Rightarrow \Delta(d\,\mathrm{Tr}\,Q_0 + \mathrm{Tr}\,R_0(\phi)) + d\,\mathrm{Tr}\,P_1 = 0\\ &\Rightarrow d(-\Delta\,\mathrm{Tr}\,Q_0 + \mathrm{Tr}\,P_1) = 0\\ &\Rightarrow \mathrm{Tr}\,P_1 = \Delta\,\mathrm{Tr}\,Q_0 + d\,\mathrm{Tr}\,Q_1 + \mathrm{Tr}\,R_1(\phi)\,. \end{aligned} \tag{2.19}$$

By induction,

$$\mathrm{Tr}\,P_i = \Delta\,\mathrm{Tr}\,Q_{i-1} + d\,\mathrm{Tr}\,Q_i + \mathrm{Tr}\,R_i(\phi), \qquad i = 1, \ldots, M\,,$$

for some polynomials Q_i and $R_i(\phi)$. So

$$\begin{aligned}\mathrm{Tr}\,P &= d\,\mathrm{Tr}\,Q_0 + \mathrm{Tr}\,R_0(\phi) + \sum_{i=1}^{i=M} \Delta\,\mathrm{Tr}\,Q_{i-1} + d\,\mathrm{Tr}\,Q_i + \mathrm{Tr}\,R_i(\phi)\\ &= (d+\Delta)\sum_{i=0}^{M-1} \mathrm{Tr}\,Q_i + \sum_{i=0}^{M} \mathrm{Tr}\,R_i(\phi) + d\,\mathrm{Tr}\,Q_M\,. \end{aligned} \tag{2.20}$$

Then

$$\delta\,\mathrm{Tr}\,P = 0 \qquad \Rightarrow \qquad \Delta d\,\mathrm{Tr}\,Q_M = 0$$

and so the technical result at the beginning of section 2.4 gives

$$d\,\mathrm{Tr}\,Q_M = (d+\Delta)\,\mathrm{Tr}\,S\,,$$

for some polynomial S, and proves the result.

3. Concluding remarks

We have considered the $\mathrm{SU}(N)$ cohomological matrix models in four, six and ten dimensions, and shown that

$$\delta\,\mathrm{Tr}\,P = 0 \Leftrightarrow \mathrm{Tr}\,P = \delta\,\mathrm{Tr}\,Q + \mathrm{Tr}\,R(\phi) \tag{3.1}$$

as long as the degree of P is less than $2N/3$.

Although the large N limit is a case of particular interest, it would also be interesting to understand what happens when N is small, or the gauge group is not $\mathrm{SU}(N)$. At present, we do not know of any counter examples to the general formula (3.1) in these cases. It would also be interesting to understand whether the result can be extended to a general gauge invariant quantity consisting of an arbitrary function of traces.

Finally we note that, in practice, the supercharge (1.7) is often deformed $\delta \to \delta_{\rm def}$ to obtain useful results from cohomological models [13, 14, 1]. A typical deformation leaves all field transformations under δ unchanged except

$$\delta_{\rm def}\, \psi_a = [\phi, X_a] + i\nu\, \epsilon_{ab} X_b\,, \tag{3.2}$$

where ν is a parameter. We point out here that it is very easy to adapt the methods of the previous section to obtain

$$\delta_{\rm def}\ {\rm Tr}\, P = 0 \Leftrightarrow {\rm Tr}\, P = \delta_{\rm def}\ {\rm Tr}\, Q + {\rm Tr}\, R(\phi)\,, \tag{3.3}$$

for N sufficiently larger than $\deg(P)$.

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